Quantum spherical spin models

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A recently introduced class of quantum spherical spin models is considered in detail. Since the spherical constraint already contains a kinetic part, the Hamiltonian need not have kinetic term. As a consequence, situations with or without momenta in the Hamiltonian can be described, which may lead to different symmetry classes. Two models that show this difference are analyzed. Both models are exactly solvable and their phase diagram is analyzed. A transversal external field leads to a phase transition line that ends in a quantum critical point. The two considered symmetries of the Hamiltonian considered give different critical phenomena in the quantum critical region. The model with momenta is argued to be analog to the large-$\mathcal{N}$ limit of an SU($\mathcal{N}$) Heisenberg ferromagnet, and the model without momenta shares the critical phenomena of an SU($\mathcal{N}$) Heisenberg antiferromagnet.

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I. INTRODUCTION

The classical spherical model was conceived by Kac. After being introduced in 1947 to Onsager's rather intricate solution of the two-dimensional (2D) Ising model, he desired to formulate a simpler spin model. As a first step he took the spins to be continuous Gaussian variables, nowadays called the Gaussian model. This had unphysical behavior at low temperatures which led Kac to consider the "spherical model." The spherical model has continuous spins that are restricted by the "spherical" constraint $\Sigma_{i=1}^{N} S_i^2 = N$, which represents the hypersphere intersecting all vertices of the hypercube sustained by the Ising spins, $S_i = \pm 1$. In the end, the spherical model is formally the same as the Ising model with a global constraint instead of a local one: the sum of spins is constrained instead of each of them. At that time the saddle point method, needed in the solution, was not widely known, and here Berlin came in, leading the celebrated joint publication on the spherical model in 1952 [1]. Kac's personal reminiscence of this history is presented in Ref. [2].

The spherical model for a ferromagnet has been considered in great detail. Actually, the paramagnetic to ferromagnetic transition is similar to an ideal Bose-Einstein condensation. Since the solution of the model is so simple and explicit, the critical behavior can be solved exactly. Critical exponents and scaling functions can be derived. In particular, the model with short range interactions exhibits $d_{\mathrm{lc}} = 2$ as the lower critical dimension; for $d \leqslant 2$ no stable ferromagnetic phase occurs. Likewise, $d_{\mathrm{uc}} = 4$ is the upper critical dimension; for $d > 4$ critical exponents take their mean-field values. These analytic results have been used to test approximations and general ideas of phase transitions for a wide range of interactions, short and long range. For a review on the classical spherical model see Ref. [3].

As said, the spherical model was introduced for its mathematical simplicity. However, Stanley [4] proved that the free energy of a model of arbitrary spin dimension ν, incorporating thus the Ising model (for spin dimension $\nu = 1$), the $x-y$ model (spin dimension $\nu = 2$) and Heisenberg model ($\nu = 3$), approaches that of the spherical model in the limit of infinite spin dimensionality $\nu \to \infty$. Hence, it gives a geometrical interpretation to the spherical model. Since various critical properties where proven to be monotonic functions of the spin dimensionality ν, the critical properties of the Heisenberg model appeared to be bounded on one side by those of the Ising model and on the other by those of the spherical model.

The spherical model for antiferromagnets was thoroughly studied by Knops. The spherical constraint imposes $\langle S_i^2 \rangle = 1$ for ferromagnets. However, this does not work for antiferromagnets because of the lack of translational invariance. To recover this, he added a second constraint; more generally, one constraint has to be added for each translationally invariant set, which in the case of antiferromagnets means each of the two sublattices. Knops found that the two constraints reduce to a unique one, provided the staggered external field vanishes. The fact that the spherical spins are scalars makes it impossible to define an order parameter that can be identified with the spontaneous staggered magnetization. To solve that and get the proper order parameter Knops used a vector version of the spherical model [5]. He also generalized Stanley's arguments to nontranslational interactions [6].

The spherical model has also been applied to disordered systems. Though, in view of Knops' finding, perhaps an infinite number of spherical constraints should be used, typically no analog of the staggered external field is applied, and one may expect that all constraints collapse into a single one. Therefore spherical spin glass models may still give insight in the physics of the problem which would be more difficult to study, for example, with Ising spins. In the case of pair couplings the exact solution exhibits no breaking of replica symmetry and the replica trick need not be used [7]. The family of p-spin spin glasses (p-spin models) [8] has been shown to exhibit one step replica symmetry breaking by studying the spherical version. For spin glasses with random pair and quartet interactions ($\{p=2\}+\{p=4\}$, "$p=2+4$"), one of us showed that an exact solution exists, exposing the full replica symmetry breaking scenario. The simplicity of

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spherical models thus may give insight in difficult problems for which otherwise no exact solution is available. For an early review on the use of the spherical model in disordered systems, see Ref. [9].

So far the discussion has been classical. The classicality can be understood in particular because the entropy diverges at low temperature as $\ln T$, just as for a classical ideal gas. Different quantum versions of the spherical model have been proposed. Obermair studied surface effects in phase transitions [10]. Identifying with a spin an operator $\hat{S}_i$, he postulated a momentum operator $\hat{\Pi}_i$ conjugate to it, $[\hat{S}_i,\hat{\Pi}_j]=i\hbar\delta_{ij}$. To get a spin dynamics, he added a kinetic term $\frac{1}{2}g\Sigma_i\hat{\Pi}_i^2$ to the Hamiltonian, but kept the constraint the same except for expressing it in operators as $\Sigma_i\langle\hat{S}_i^2\rangle=N$. In this case, the kinetic term may be understood as the kinetic energy of rigid rotors. The model remains exactly solvable. Many others have therefore used this quantization in the study of spin glasses [11], systems with multispin random interactions (p-spin glasses) [12], or the study of quantum phase transitions [13].

One of us presented in 1995 a different quantum approach to cure the low temperature behavior [14]. In a Trotter approach to the partition sum, the first step is to take as spherical constraint $\Sigma_i\Sigma_i^*\Sigma_i=N\tilde{\sigma}/\hbar^2$, where $\tilde{\sigma}$ is a constant that need not be unity, and Σ_i is the complex parameter characterizing the coherent state associated with the bosonic annihilation operator $\hat{\Sigma}_i=[\hat{S}_i/\hbar+i\hat{\Pi}_i]/\sqrt{2}$. Hence, in this approach the momentum appears in the constraint. Indeed, this constraint may also be written $\Sigma_i\langle\hat{S}_i^2\rangle+\hbar^2\langle\hat{\Pi}_i^2\rangle=2N\tilde{\sigma}$. As a second step, momenta dependent Hamiltonians were considered, by replacing $J_{ij}S_iS_j\rightarrow J_{ij}\hat{\Sigma}_i^\dagger\hat{\Sigma}_j$. Later [15], the same formalism was applied to the p-spin glass model and was compared with its Ising counterpart. In spite of the simplicity of the system and its solubility, the resulting phase diagram shows very interesting critical phenomena.

Since the momenta are present in the constraint one may also study situations where they do not appear in the Hamiltonian. Below we will consider two Hamiltonians with nearest neighbor ferromagnetic interactions. We will see that the presence or absence of momenta can change the symmetries of the action giving rise to different critical phenomena in the quantum region. The resulting action may be invariant under unitary transformations or under orthogonal ones, while in Obermair's approach only the latter is possible. In the final section we show that one of these spherical spin models relates to a quantum ferromagnet and the other to a quantum antiferromagnet.

The two different quantum versions of the model have the same quantization rule $[\hat{S}_i,\hat{\Pi}_j]=i\hbar\delta_{i,j}$. Both of them cure the problem of the entropy, it remains positive and, for temperature going to zero, goes to zero as a power law. Vojta, [13] following Stanley's arguments, found that Obermair's quantization gives a free energy that is identical to the large-n limit of the O(n) nonlinear sigma model. Therefore it describes rotors instead of Heisenberg spins. Nieuwenhuizen, conversely, gave indications that his version had a behavior closer to Heisenberg spins, as having in the case of free spins a gap between the ground and the first excited state scaling with the field at small fields.

The aim of the present paper is to point out the differences between these two models. We study two Hamiltonians using Nieuwenhuizen's spherical constraint. The first one was introduced in Ref. [14] and we find it to be analogous to the large-$\mathcal{N}$ limit of a SU($\mathcal{N}$) Heisenberg ferromagnet. In the second Hamiltonian studied, no momenta are present; momenta only appear in the formalism through the spherical constraint. In this case, the same critical phenomena as in the large-$\mathcal{N}$ limit of a SU($\mathcal{N}$) Heisenberg antiferromagnet is found, which can be described by an O($\mathcal{N}$) nonlinear σ model which, in turn, is analogous to Obermair's model.

The paper is organized as follows. In Sec. II the classical spherical model is reviewed and the way to quantize it is discussed. The differences between the two quantum spherical constraints are pointed out. In Sec. III, the path integral formalism is introduced to calculate the partition function of a quantum spherical model. In Sec. IV this formalism is used to solve the thermodynamics of a ferromagnetic quantum spherical model with nearest neighbor interaction and the critical phenomena is studied in detail. At the finite temperature phase transition, the critical exponents remain the same as the classical ones. In Sec. V a Hamiltonian with the same couplings but without momenta is considered. The critical exponents are found to be the same as in Obermair's model. After that, in Sec. VI a generalization of the two types of Hamiltonians presented here is given and the limit of SU($\mathcal{N}$) Heisenberg spins is argued to give the same critical behavior as this quantum version of spherical model. Finally some conclusions are drawn.

II. CLASSICAL SPHERICAL MODEL

The spherical constraint was conceived as a relaxation of the Ising constraint. Indeed, Ising spins, $S_i\equiv s_{i,z}=\pm\frac{1}{2}\hbar$, obviously satisfy it. Adjusting the coefficients from the original version it may be written as

$$\frac{1}{2}\sum_{i=1}^{N}S_i^2=N\sigma, \tag{1}$$

with $\sigma=\hbar^2/8$ having dimension (Js)^2. The Berlin-Kac spherical model is defined by the partition sum

$$\begin{aligned} Z&=\int DS\, e^{-\beta H}\delta\left(\frac{1}{2}\sum_{i=1}^{N}S_i^2-N\sigma\right)\\ &=\int DS\int_{-i\infty}^{i\infty}\frac{d\tilde{\mu}}{2\pi i}e^{-\beta H-(1/2)\tilde{\mu}\Sigma_{i=1}^{N}S_i^2+\tilde{\mu}N\sigma}, \end{aligned} \tag{2}$$

where

$$DS=\prod_i\int_{-\infty}^{\infty}dS_i. \tag{3}$$

A. Vector spherical spins

For vector spins the generalization of Eq. (1) in the case of m spin dimensions reads

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{a=1}^{m}(S_i^a)^2=Nm\sigma. \tag{4}$$

It is worth mentioning that the spin dimensionality in Eq. (4) is not related to the approach of Stanley, who started with vector spins and ended up with scalar spherical spins. We only introduce vector spherical spins to avoid the restriction scalar spins have. We benefit from the fact that the vector character allows us to study the behavior in a transverse field. A similar step allowed Knops to define a proper order parameter for the antiferromagnetic spherical model [5].

B. Quantization

It is natural to consider the S_i analogous to position variables of harmonic oscillators. In quantum mechanics they become hermitian operators $\hat{S}_i$ with the dimension of $\hbar$, Js. The conjugate momentum operator $\hat{\Pi}_i^a$ is dimensionless and postulated to satisfy the commutation relation

$$[\hat{S}_i^a,\hat{\Pi}_j^b]=i\hbar\,\delta_{i,j}\delta_{a,b}\,. \tag{5}$$

As for harmonic oscillators, this allows us to define creation and annihilation operators

$$\hat{\Sigma}_i^{a\,\dagger}=\frac{1}{\hbar\sqrt{2}}\hat{S}_i^a-\frac{i}{\sqrt{2}}\hat{\Pi}_i^a,\quad \hat{\Sigma}_i^a=\frac{1}{\hbar\sqrt{2}}\hat{S}_i^a+\frac{i}{\sqrt{2}}\hat{\Pi}_i^a \tag{6}$$

satisfying the commutation relation

$$[\hat{\Sigma}_i^a,\hat{\Sigma}_j^{b\,\dagger}]=\delta_{i,j}\delta_{a,b}\,. \tag{7}$$

C. Spherical constraint on the length of the total spin

There is some freedom to choose the spherical constraint, which amounts to describing different physical situations. The standard quantum constraint considered in literature is just the quantized version of the mean of Eq. (4),

$$\text{constraint 1:}\quad \frac{1}{2}\sum_{i,a}\langle(\hat{S}_i^a)^2\rangle=Nm\sigma, \tag{8}$$

where $\langle\cdots\rangle$ denotes the quantum expectation value. Obermair took as the quantum Hamiltonian the classical $H(\mathbf{S})$ with spins replaced by operators, and added the kinetic term that one expects for physical rotors,

$$\hat{H}(\hat{\mathbf{S}},\hat{\Pi}i)=\frac{1}{2}g\sum_i\hat{\Pi}_i^2+H(\hat{\mathbf{S}}), \tag{9}$$

where g^{-1} is the rotor's moment of inertia. An effective Hamiltonian which includes the constraint can be derived with a Lagrange multiplier. One ends up with

$$\hat{H}_{\rm tot}=\frac{1}{2}g\sum_i\hat{\Pi}_i^2+H(\hat{\mathbf{S}})+\mu(t)\left[\frac{1}{2}\sum_{i,a}(\hat{S}_i^a)^2-Nm\sigma\right], \tag{10}$$

where μ is the Lagrange multiplier that enforces the constraint. In equilibrium its value is given by the equation of the spherical constraint $\partial F/\partial\mu=0$. The dynamics is now fixed by the Heisenberg equations of motion,

$$\frac{\mathrm{d}\hat{S}_i^a}{\mathrm{d}t}=i[\hat{H}_{\rm tot}(t),\hat{S}_i^a(t)]=g\hat{\Pi}_i^a(t), \tag{11}$$

$$\frac{\mathrm{d}\hat{\Pi}_i^a}{\mathrm{d}t}=i[\hat{H}_{\rm tot}(t),\hat{\Pi}_i^a(t)]=-\frac{\partial\hat{H}}{\partial\hat{S}_i^a}-\mu(t)\hat{S}_i^a(t). \tag{12}$$

where the real parameter μ has to be taken time-dependent in order to satisfy the soft constraint (8) at each instant. It is clear that a nonzero g is needed to get any spin dynamics. Combining the two equations one has

$$\frac{1}{g}\frac{\mathrm{d}^2\hat{S}_i^a}{\mathrm{d}t^2}=-\frac{\partial\hat{H}}{\partial\hat{S}_i^a}-\mu(t)\hat{S}_i^a(t). \tag{13}$$

It is worth remarking that no energy budget is involved in the spherical constraint,

$$\langle\hat{H}_{\rm tot}\rangle=\langle\hat{H}\rangle. \tag{14}$$

D. Spherical constraint on the number of spin quanta

In 1995 one of us had proposed a constraint that fixes the number of quanta [14]. In a path integral approach it was assumed that the c numbers Σ_i^a, which characterize a coherent state, satisfy at each timestep

$$\text{constraint 2}':\quad \sum_{i,a}\Sigma_i^{a*}\Sigma_i^a=Nm\frac{\tilde{\sigma}}{\hbar^2}. \tag{15}$$

It is to be expected that this is equivalent to

$$\text{constraint 2:}\quad \sum_{i,a}\langle\hat{\Sigma}_i^{a\dagger}\hat{\Sigma}_i^a\rangle=\sum_{i,a}\langle n_i^a\rangle=Nm\frac{\sigma}{\hbar^2}. \tag{16}$$

We shall show below that this is indeed the case, and the relation, $\tilde{\sigma}=\sigma+\hbar^2$, is derived in Eq. (38). This constraint includes the momenta as can be seen by writing it in the form

$$\text{constraint 2:}\quad \frac{1}{2}\sum_{i,a}(\langle\hat{S}_i^{a\,2}\rangle+\hbar^2\langle\hat{\Pi}_i^{a\,2}\rangle)=Nm\left(\sigma+\frac{\hbar^2}{2}\right) \tag{17}$$

For a Hamiltonian $\hat{H}(\hat{\mathbf{S}},\hat{\Pi})$ that may, but need not, depend explicitly on the momenta. The effective spherical Hamiltonian is

$$\hat{H}_{\text{tot}}=\hat{H}(\hat{\mathbf{S}},\hat{\Pi})+\frac{1}{2}\mu\sum_{i,a}\left[(\hat{S}_i^a)^2+\hbar^2(\hat{\Pi}_i^a)^2\right]-Nm\mu\left(\sigma+\frac{\hbar^2}{2}\right). \tag{18}$$

Now, the situation where the Hamiltonian does not depend explicitly on the momenta (no kinetic term), $\hat{H}(\hat{\mathbf{S}},\hat{\Pi})\rightarrow\hat{H}(\hat{\mathbf{S}})$, still leads to sensible dynamics, since the constraint already depends on the momenta. Different constraints describe different physics. However, at high temperatures one expects the differences to become small.

Equation (11) now brings $\hat{\Pi}_i^a=(\mathrm{d}\hat{S}_i^a/\mathrm{d}t)/\mu(t)$, Eq. (12) remains the same. They may be combined together into a second order equation for the spin operators,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu(t)}\frac{\mathrm{d}\hat{S}_i^a}{\mathrm{d}t}\right)=-\frac{\partial\hat{H}}{\partial\hat{S}_i^a}-\mu(t)\hat{S}_i^a(t). \tag{19}$$

In the remaining of this paper we will simplify the notation by taking units in which $\hbar=1$.

E. Comparison of the two constraints

The main difference between the two constraints is obviously the presence or absence of momenta. In the second case, Eq. (16), the spherical constraint can carry all the dynamics of the model. On the contrary, using the first constraint, Eq. (8), a kinetic term, with an external parameter g, has to be added to the Hamiltonian [10]. This parameter determines the strength of quantum fluctuations; the classical model can be recovered for $g=0$. This fact makes models with the first constraint describe quantum rotors, as was pointed out in Ref. [13]. The first constraint, Eq. (8), brings actions which are invariant under orthogonal transformations. Conversely, using the second constraint, Eq. (16), the choice of Hamiltonian can bring symmetry under unitary transformations or orthogonal ones depending on the question whether the Hamiltonian contains momenta or not. Hamiltonians with unitary transformation symmetry yield free energies analogous to the large $\mathcal{N}$ limit of the generalization of SU(2) Heisenberg spins to SU($\mathcal{N}$). Hamiltonians with orthogonal transformation symmetry share the critical phenomena with the large $\mathcal{N}$ limit of O($\mathcal{N}$) nonlinear sigma model and describe therefore quantum rotors as occurs by using the first constraint, Eq. (8).

Each of the symmetries belong in different universality classes in the quantum regime, yet classical critical phenomena are always the same as in the classical model, consistent with the expectation that quantum effects do not lead to qualitative changes at finite temperatures. We will see that the dynamical critical exponent z is different in both symmetries, causing the difference in critical exponents at the quantum critical point as was pointed out in Ref. [16].

III. PATH INTEGRALS

In this section we explain, following Ref. [15], how to add the spherical constraint to a quantum Hamiltonian using the path integral formalism for models with the second constraint, Eq. (16). In second quantization the spins are given a bosonic algebra. In the path integral the boson coherent state representation is used for the spins (for a review of path integrals and coherent states, see, e.g., Ref. [17] and for a complete study of coherent states see, e.g., [18]).

A. Bosonic coherent state representation for a single oscillator

Fock space is the Hilbert space of states labeled by the number of oscillator quanta. Coherent states are defined as the eigenstates of the annihilator operator $\hat{a}$. Then it can be proved that for a system with many particles

$$|\phi\rangle=e^{\Sigma_\alpha\phi_\alpha\hat{a}^\dagger_\alpha}|0\rangle=\prod_\alpha\left\{\sum_{n_\alpha}\frac{(\phi_\alpha\hat{a}^\dagger{}_\alpha)^{n_\alpha}}{n_\alpha!}\right\}|0\rangle \tag{20}$$

is a coherent state, where $|0\rangle$ is the vacuum representation in Fock's space, and α stands for each state for any particle of the system. Indeed, because of the identity $\hat{a}_\alpha(\hat{a}^\dagger_\alpha)^{n_\alpha}=n_\alpha(\hat{a}^\dagger_\alpha)^{n_\alpha-1}+(\hat{a}^\dagger_\alpha)^{n_\alpha}\hat{a}_\alpha$, it holds that $\hat{a}_\alpha|\phi\rangle=\phi_\alpha|\phi\rangle$. The scalar product of two coherent states gives

$$\langle\phi|\phi'\rangle=e^{\Sigma_\alpha\phi^*_\alpha\phi'_\alpha}. \tag{21}$$

A crucial property of the coherent states is that they form an overcomplete set of states. Any vector in Fock space can then be expanded in terms of coherent states. This is expressed by the closure relation [17]

$$\int\prod_\alpha\frac{d\mathrm{Im}(\phi_\alpha)d\mathrm{Re}(\phi_\alpha)}{\pi}e^{-\Sigma_\alpha\phi^*_\alpha\phi_\alpha}|\phi\rangle\langle\phi|=\mathbf{1} \tag{22}$$

where the measure in the integral comes from gaussian integration with complex variables and the exponential term is due to the fact that coherent states are not normalized. Let us check that Eq. (22) is indeed a representation of the identity of Fock space. We insert it in the left hand side of Eq. (21) and we get

$$\begin{aligned}\langle\phi|\mathbf{1}|\phi'\rangle&=\int\prod_\alpha\frac{d\mathrm{Im}(\psi_\alpha)d\mathrm{Re}(\psi_\alpha)}{\pi}e^{-\Sigma_\alpha\psi^*_\alpha\psi_\alpha}\langle\phi|\psi\rangle\langle\psi|\phi'\rangle\\&=\int\prod_\alpha\frac{d\mathrm{Im}(\psi_\alpha)d\mathrm{Re}(\psi_\alpha)}{\pi}e^{\Sigma_\alpha-(\psi^*_\alpha\psi_\alpha-\phi^*_\alpha\psi_\alpha-\psi^*_\alpha\phi'_\alpha)}\\&=e^{\Sigma_\alpha\phi^*_\alpha\phi'_\alpha}\end{aligned} \tag{23}$$

which indeed is the right hand side of Eq. (21).

The partition function of any quantum system $Z=\mathrm{tr}[e^{-\beta H(\hat{a}^\dagger,\hat{a})}]$ can be computed by the Trotter approach. The exponential has the same form as a time evolution operator in imaginary time. Thus it is possible to create a path integral over closed paths. The procedure is to split the exponential in a product of M equal terms. Between each pair of them a representation of the identity, Eq. (22), is inserted. The partition sum then has the following shape:

$$Z=\mathrm{tr}\{(e^{-\epsilon H(\hat{a}^\dagger,\hat{a})})^M\}$$
$$=\mathrm{tr}\{e^{-\epsilon H(\hat{a}^\dagger,\hat{a})}\mathbf{1}e^{-\epsilon H(\hat{a}^\dagger,\hat{a})}\mathbf{1}\cdots\mathbf{1}\,e^{-\epsilon H(\hat{a}^\dagger,\hat{a})}\}, \quad (24)$$

where $\epsilon=\beta/M$ and each $\mathbf{1}$ is an identity operator. Each of these identities is given an index; they represent the steps the system passes through in a discretized path. By using the identity defined in Eq. (22) the following matrix element is needed:

$$\langle\phi_j|e^{-\epsilon H(\hat{a}^\dagger,\hat{a})}|\phi_{j-1}\rangle. \quad (25)$$

Provided the Hamiltonian is normal ordered, the outcome is [17]

$$\langle\phi_j|e^{-\epsilon H(\hat{a}^\dagger,\hat{a})}|\phi_{j-1}\rangle\approx\langle\phi_j|1-\epsilon H(\hat{a}^\dagger,\hat{a})|\phi_{j-1}\rangle$$
$$=e^{\phi_j^*\cdot\phi_{j-1}}[1-\epsilon H(\phi_j^*,\phi_{j-1})]$$
$$=e^{\phi_j^*\cdot\phi_{j-1}-\epsilon H(\phi_j^*,\phi_{j-1})}+\mathcal{O}(\epsilon^2). \quad (26)$$

Correction terms can be neglected in the limit $M\to\infty$ [17]. Each identity brings an integral at each time step. These integrals cover any path between its initial and its final state. The trace will finally tie the ends giving a closed path. The partition function finally reads

$$Z=\int_{\phi_\alpha(\beta)=\phi_\alpha(0)}D(\phi_\alpha^*(\tau)\phi_\alpha(\tau))\times\exp\left\{\sum_{\tau=0}^{\beta}d\tau\left[\phi^*(\tau)\frac{d\phi(\tau)}{d\tau}+H(\phi^*(\tau),\phi(\tau-d\tau))\right]\right\}, \quad (27)$$

where the subindex of the integral reflects the trace structure of the partition function since it gives a closed path integral; τ stands for the imaginary time step, so $\phi(\tau)=\phi_i$; $d\tau$ is the imaginary time difference between steps, so $\phi(\tau-d\tau)=\phi_{i-1}$; and

$$\frac{d\phi(\tau)}{d\tau}=\frac{\phi(\tau)-\phi(\tau-d\tau)}{d\tau}=\frac{\phi_i-\phi_{i-1}}{\beta/M}. \quad (28)$$

Despite the fact that the nomenclature used in these formulas suggests a continuous time, it should always be understood as being discrete. The limit $M\to\infty$ should always be taken at the end of the calculations, otherwise some indeterminacies may arise. Continuous notation is used nevertheless because it is more compact.

B. Coherent state representation for spherical spins

We can deal with spherical spins using almost the same approach. The operator $\hat{a}_i$ is identified with $\hat{\Sigma}_i^a$, where the index a denotes the spin vector direction, and the corresponding fields ϕ_i are denoted as Σ_i^a. We remind that the spherical constraint we use is the one defined in Eq. (16).

In order to impose this constraint in the path integral formalism, the identity definition Eq. (22) is modified to adopt to the spherical case, in a way inspired by Ref. [15]: one restricts the path integral to states which exactly satisfy the constraint by employing the truncated identity

$$\mathbf{1}\to\mathbf{1}_{spherical}\equiv C\int\prod_{ia}\frac{d\mathrm{Im}(\Sigma_i^a)d\mathrm{Re}(\Sigma_i^a)}{\pi}e^{-\mathbf{\Sigma}^*\cdot\mathbf{\Sigma}}|\mathbf{\Sigma}\rangle\langle\mathbf{\Sigma}|\,\delta(\hat{\mathbf{n}}-Nm\sigma) \quad (29)$$

where the number operator $\hat{\mathbf{n}}$,

$$\hat{\mathbf{n}}=\sum_{i,a}\hat{\Sigma}_i^{a\dagger}\hat{\Sigma}_i^a, \quad (30)$$

counts the total number of spin quanta. We insert

$$\delta(\hat{\mathbf{n}}-Nm\sigma)=\int_{-\infty}^{\infty}\frac{\epsilon\mathrm{d}\tilde{\mu}}{2\pi}e^{-i\epsilon\tilde{\mu}(\hat{\mathbf{n}}-Nm\sigma)}$$
$$=\int_{-i\infty}^{i\infty}\frac{\epsilon\mathrm{d}\mu}{2\pi i}e^{-\epsilon\mu(\hat{\mathbf{n}}-Nm\sigma)}, \quad (31)$$

where $\mu=i\tilde{\mu}$ is imaginary. (Strictly speaking, we should insert a Kronecker-δ function, rather than the Dirac-δ, but for large N this amounts to the same.) Repeating the same procedure with this new identity we get

$$Z=\int_{\Sigma(\beta)=\Sigma(0)}D\mu D\mathbf{\Sigma}^*D\mathbf{\Sigma}\exp(-\mathbf{A}), \quad (32)$$

with the action

$$\mathbf{A}=\sum_{\tau=0}^{\beta}d\tau\Big[\mathbf{\Sigma}^*(\tau)\cdot\frac{d\mathbf{\Sigma}(\tau)}{d\tau}+\mu(\tau)(\mathbf{\Sigma}^*(\tau)\cdot\mathbf{\Sigma}(\tau-d\tau)-Nm\sigma)+H(\mathbf{\Sigma}^*(\tau),\mathbf{\Sigma}(\tau-d\tau))\Big] \quad (33)$$

and integration measures defined as

$$\int D\mathbf{\Sigma}^*D\mathbf{\Sigma}=\prod_{ia\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\mathrm{Im}(\Sigma_i^a(\tau))d\mathrm{Re}(\Sigma_i^a(\tau))}{\pi} \quad (34a)$$

$$\int D\mu=C\prod_{\tau}\int_{-i\infty}^{i\infty}\frac{\epsilon d\mu(\tau)}{2\pi i} \quad (34b)$$

where $\mu(\tau)$ is the Lagrange multiplier introduced to impose the spherical constraint and the prefactor $C^{(M)}$ is added to ensure, if needed, a proper normalization. Details on this factor are given in Ref. [15].

It should be noted that the particle number operator in the definition of the identity, Eq. (29), will be surrounded, as is the case for the Hamiltonian, by spin operators on different timesteps; therefore its creation and annihilation operators will also be projected on different timesteps, $\hat{\Sigma}_i^{a\,\dagger}\hat{\Sigma}_i^a \to \Sigma_i^{a\,*}(\tau)\Sigma_i^a(\tau-\mathrm{d}\tau)$. In Refs. [14,15] the spherical constraint was slightly different from the one presented here. The proposal was to take the constraint not in terms of the particle number operator but in terms of its generating variables (which are c numbers), at every imaginary timestep,

$$\mathbf{\Sigma}^*\cdot\mathbf{\Sigma}=\sum_{i=1}^{N}\sum_{a=1}^{m}\Sigma_i^{a*}(\tau)\Sigma_i^a(\tau)=Nm\tilde{\sigma} \tag{35}$$

so they acquire the same time-index. The two actions then differ only in the timestep projection of the spherical constraint, so with this constraint one obtained $\mathbf{\Sigma}^*(\tau)\cdot\mathbf{\Sigma}(\tau)$ rather than $\mathbf{\Sigma}^*(\tau)\cdot\mathbf{\Sigma}(\tau-d\tau)$. The difference that this brings can be seen as follows. Starting from Eq. (32) we want to have two operators projected at the same time. To achieve this, it turns out that we must exchange the order of the operators, $\Sigma^\dagger\Sigma=\Sigma\Sigma^\dagger-1$. The term $\Sigma\Sigma^\dagger$ can be projected at a single time as one can see following Eq. (26). For a single component spin at timestep j the relevant matrix element is

$$\begin{aligned}
&\langle\Sigma_{j+1}|e^{-\epsilon\hat{H}}e^{-\epsilon\mu\hat{\Sigma}^\dagger\hat{\Sigma}}e^{-\epsilon\hat{H}}|\Sigma_{j-1}\rangle\\
&\quad=\langle\Sigma_{j+1}|e^{-\epsilon\hat{H}}(1+\epsilon\mu-\epsilon\mu\hat{\Sigma}\hat{\Sigma}^\dagger)e^{-\epsilon\hat{H}}|\Sigma_{j-1}\rangle\\
&\quad=(1+\epsilon\mu)\langle\Sigma_{j+1}|e^{-\epsilon\hat{H}}|\Sigma_j\rangle\langle\Sigma_j|e^{-\epsilon\hat{H}}|\Sigma_{j-1}\rangle\\
&\qquad-\epsilon\mu\langle\Sigma_{j+1}|e^{-\epsilon\hat{H}}\hat{\Sigma}|\Sigma_j\rangle\langle\Sigma_j|\hat{\Sigma}^\dagger e^{-\epsilon\hat{H}}|\Sigma_{j-1}\rangle\\
&\quad=(1+\epsilon\mu-\epsilon\mu\Sigma_j^*\Sigma_j)\langle\Sigma_{j+1}|e^{-\epsilon\hat{H}}|\Sigma_j\rangle\langle\Sigma_j|e^{-\epsilon\hat{H}}|\Sigma_{j-1}\rangle\\
&\quad=e^{\epsilon\mu-\epsilon\mu\Sigma_j^*\Sigma_j}\langle\Sigma_{j+1}|e^{-\epsilon\hat{H}}|\Sigma_j\rangle\langle\Sigma_j|e^{-\epsilon\hat{H}}|\Sigma_{j-1}\rangle
\end{aligned} \tag{36}$$

Thus a factor $e^{\epsilon\mu-\epsilon\mu\Sigma_i^{a\,*}(\tau)\Sigma_i^a(\tau)}$ comes for each spin operator $\hat{\Sigma}_i^a$ at each timestep. This leads to the spherical constraint

$$\mathbf{1}_{spherical}\equiv C\int\prod_{ia}\frac{d\mathrm{Im}(\Sigma_i^a)d\mathrm{Re}\,(\Sigma_i^a)}{\pi}e^{-\mathbf{\Sigma}^*\cdot\mathbf{\Sigma}}|\mathbf{\Sigma}\rangle \times\langle\mathbf{\Sigma}|\,\delta\!\left(\sum_{i,a}\Sigma_i^{a\,*}\Sigma_i^a-Nm-Nm\sigma\right) \tag{37}$$

that with definition Eq. (35) should be compared to Eq. (29). In words, the spherical constraint can indeed be taken on the coherent state variables as in Eq. (35) or in Ref. [14], provided one makes the identification $\tilde{\sigma}=\sigma+1$, or, restoring units,

$$\tilde{\sigma}=\sigma+\hbar^2. \tag{38}$$

In Eq. (48) we will verify that with this identification the two approaches indeed yield the same free energy.

It is worth remarking that we imposed the spherical constraint strictly, no thermal average has been performed. In the following section μ will be integrated over by the method of steepest descends, a procedure that allows the particle number to fluctuate; therefore the satisfiability of the constraint remains only in average.

IV. FERROMAGNETIC HAMILTONIANS WITH CREATION AND ANNIHILATION OPERATORS

Using the formalism described in Sec. III we can study the Hamiltonian

$$\begin{aligned}
H(\hat{\mathbf{\Sigma}}^\dagger,\hat{\mathbf{\Sigma}})&=-\sum_{i\neq j}J_{ij}\hat{\mathbf{\Sigma}}_i^\dagger\hat{\mathbf{\Sigma}}_j-\sum_i\Gamma_i\frac{(\hat{\mathbf{\Sigma}}_i^\dagger+\hat{\mathbf{\Sigma}}_i)}{\sqrt{2}}\\
&=-\frac{1}{2}\sum_{i\neq j}J_{ij}(\hat{\mathbf{S}}_i\hat{\mathbf{S}}_j+\hat{\mathbf{\Pi}}_i\hat{\mathbf{\Pi}}_j)-\sum_i\Gamma_i\hat{\mathbf{S}}_i
\end{aligned} \tag{39}$$

where in the second equality we inserted in Eq. (6). The $i\hat{S}_i\hat{\Pi}_j$ cancelled since we assumed symmetric couplings, $J_{ij}=J_{ji}$. Obviously, the momentum operators do occur in this expression. The couplings J_{ij} can in principle express any kind of interaction, ferromagnetic, antiferromagnetic, spin glass, etc. The Γ_i represent an external field, that can be constant, variable, random etc. Later on, we will focus on ferromagnetic couplings in the presence of constant magnetic field. This Hamiltonian without the external magnetic field is symmetric under unitary transformations, a fact that will determine the critical behavior.

The first step to get the partition function is to diagonalize the couplings,

$$\mathbf{\Sigma}_i(\tau)=\sum_\lambda\mathbf{\Sigma}_\lambda(\tau)e_i^\lambda,$$

$$\mathbf{\Sigma}_\lambda(\tau)=\sum_i\mathbf{\Sigma}_i(\tau)e_i^\lambda, \tag{40}$$

where e_i^λ is the normalized eigenvector of the coupling matrix J_{ij}.

Keeping in mind its ill definedness, we may write the partition function sum as a continuum expression,

$$\begin{aligned}
Z=\int D\tau D\Sigma D\Sigma^*\exp\Bigg\{-\int d\tau\sum_{a,\lambda}\Bigg[\Sigma_\lambda^{a*}(\tau)\frac{d\Sigma_\lambda^a(\tau)}{d\tau}\\
+\mu(\tau)(\Sigma_\lambda^{a*}(\tau)\Sigma_\lambda^a(\tau-d\tau)-Nm\sigma)-J_\lambda\Sigma_\lambda^{a*}(\tau)\Sigma_\lambda^a(\tau\\
-d\tau)-\frac{1}{\sqrt{2}}\Gamma_\lambda(\Sigma_\lambda^{a*}(\tau)+\Sigma_\lambda^a(\tau-d\tau))\Bigg]\Bigg\}.
\end{aligned} \tag{41}$$

In discrete notation, the action of Eq. (32) reads

$$\mathbf{A}=\sum_j \epsilon\left\{\frac{1}{\epsilon}\sum_{a,\lambda}\left[\Sigma^{a*}_{\lambda,j}\Sigma^a_{\lambda,j}-\Sigma^{a*}_{\lambda,j}\Sigma^a_{\lambda,j-1}\right]+\mu(j\epsilon)\right.$$
$$\times\left[\sum_{a,\lambda}\Sigma^{a*}_{\lambda,j}\Sigma^a_{\lambda,j-1}-Nm\sigma\right]-\sum_{a,\lambda}J_\lambda\Sigma^{a*}_{\lambda,j}\Sigma^a_{\lambda,j-1}$$
$$\left.-\sum_{a,\lambda}\Gamma_\lambda\frac{(\Sigma^{a*}_{\lambda,j}+\Sigma^a_{\lambda,j-1})}{\sqrt{2}}\right\} \tag{42}$$

where $\epsilon=d\tau$ is the imaginary time step, j the time index and $\Gamma_\lambda=\Sigma_i\Gamma_i e^i_\lambda$ is the field in the basis of eigenvectors of J_{ij}. Collecting all terms we have

$$Z=\int D\mu\prod_{\lambda,a}\left\{\int\prod_j\left(\frac{d\Sigma^{a*}_{\lambda,j}d\Sigma^a_{\lambda,j}}{2\pi i}\right)\exp\left[-\sum_{ij}\Sigma^{a*}_{\lambda,i}\mathbf{B}_{ij}\Sigma^a_{\lambda,j}\right.\right.$$
$$\left.\left.+\sum_j\epsilon\Gamma_\lambda\frac{(\Sigma^{a*}_{\lambda,j}+\Sigma^a_{\lambda,j-1})}{\sqrt{2}}\right]\right\}e^{\Sigma_j Nm\sigma\epsilon\mu(j\epsilon)}, \tag{43}$$

where $\mathbf{B_{ij}}=\delta_{ij}-[1+\epsilon J_\lambda-\epsilon\mu(j\epsilon)]\delta'_{i,j+1}$; here the prime stands for the fact that $\delta'_{1,M+1}\equiv 1$ due to the trace structure of the partition function. We can now integrate over the spins

$$Z=\int D\mu\exp\left[\sum_{\lambda,a}\left\{-m\ln\det\mathbf{B}_{ij}+\frac{\epsilon^2\Gamma^2_\lambda}{2}\sum_{ij}\mathbf{B}^{-1}_{ij}\right.\right.$$
$$\left.\left.+m\sigma\epsilon\sum_j\mu(j\epsilon)\right\}\right]. \tag{44}$$

As usual, in thermodynamics, one-time quantities like $\mu(\tau)$ can be taken independent of τ. We will employ this simplification throughout the rest of this paper. The determinant and the matrix inversion can then be performed [17]. Integrating over μ by the saddle point method we obtain

$$\beta F=-m\sigma\beta\mu+\frac{1}{N}\sum_{\lambda,a}\left\{\ln(1-a_\lambda)-\frac{M\epsilon^2\Gamma^2_\lambda}{2(1-a_\lambda)}\right\}, \tag{45}$$

where $a_\lambda=1-\epsilon(\mu-J_\lambda)$. Sending $M\to\infty$ we finally get

$$\beta F=-\beta\mu m\sigma+\frac{m}{N}\sum_\lambda\left\{\ln(1-e^{-\beta(\mu-J_\lambda)})-\frac{\beta\Gamma^2_\lambda}{2(\mu-J_\lambda)}\right\}$$
$$=-\beta\mu m\left(\sigma+\frac{1}{2}\right)+m\int dJ_\lambda\rho(J_\lambda)\left\{\ln\left[2\sinh\left(\frac{\beta}{2}(\mu\right.\right.\right.$$
$$\left.\left.\left.-J_\lambda)\right)\right]-\frac{\beta\Gamma^2_\lambda}{2(\mu-J_\lambda)}\right\}, \tag{46}$$

where in the last equality we have assumed that the couplings satisfy $(1/N)\Sigma_\lambda J_\lambda=0$. The saddle point equation reads

$$\sigma+1=\frac{1}{N}\sum_\lambda\left\{\frac{1}{1-e^{-\beta(\mu-J_\lambda)}}+\frac{\Gamma^2_\lambda}{2(\mu-J_\lambda)^2}\right\}$$
$$=\int dJ_\lambda\rho(J_\lambda)\left\{\frac{1}{1-e^{-\beta(\mu-J_\lambda)}}+\frac{\Gamma^2_\lambda}{2(\mu-J_\lambda)^2}\right\}. \tag{47}$$

The sums over the different eigenvalues of the coupling matrix have been changed into integrals. Each J_λ has a weight in this integral given by $\rho(J_\lambda)$. The actual form for this weight function will depend on the type of couplings. A set of weight functions for ferromagnets in different cubic lattices can be found in Ref. [3], and for spin glasses with long range interactions in Refs. [15,7].

In Ref. [15], where the spherical constraint used was the one in Eq. (35), the matrix **B** was different, namely, $\mathbf{B_{ij}}=[1+\epsilon\mu(j\epsilon)]\boldsymbol{\delta}_{ij}-(1+\epsilon J_\lambda)\boldsymbol{\delta}'_{i,j+1}$. Then Eq. (46) reads

$$\beta F=-\beta\mu m\tilde{\sigma}+\frac{m}{N}\sum_\lambda\left\{\ln(e^{\beta\mu}-e^{\beta J_\lambda})-\frac{\beta\Gamma^2_\lambda}{2(\mu-J_\lambda)}\right\}$$
$$=-\beta\mu m\left(\tilde{\sigma}-\frac{1}{2}\right)+m\int dJ_\lambda\rho(J_\lambda)\left\{\ln\left[2\sinh\left(\frac{\beta}{2}(\mu\right.\right.\right.$$
$$\left.\left.\left.-J_\lambda)\right)\right]-\frac{\beta\Gamma^2_\lambda}{2(\mu-J_\lambda)}\right\} \tag{48}$$

confirming that the already found shift $\tilde{\sigma}=\sigma+1$, see Eq. (38), indeed brings the same value for the free energy.

At large temperatures these equations reduce to

$$\beta F=-\beta\mu m\tilde{\sigma}+m\int dJ_\lambda\rho(J_\lambda)\left\{\ln\beta(\mu-J_\lambda)-\frac{\beta\Gamma^2_\lambda}{2(\mu-J_\lambda)}\right\}, \tag{49}$$

$$\tilde{\sigma}=\int dJ_\lambda\rho(J_\lambda)\left\{\frac{T}{\mu-J_\lambda}+\frac{\Gamma^2_\lambda}{2(\mu-J_\lambda)^2}\right\}. \tag{50}$$

Apart from a factor two, these are exactly the equations of the classical spherical model, see, e.g., Ref. [3]. This factor 2 arises because the momenta double the degrees of freedom, see, e.g., Ref. [14]. Near the phase transition they are already approximate, but the transition stays within the same universality class.

A. Ferromagnetic couplings with transversal field in *d* dimensions

In this section we will use the results given in the preceding section for the concrete case of ferromagnetism couplings with uniform transversal field. The Hamiltonian in this case differs from the one before Eq. (39) in the fact that the couplings only act in the *z* direction while the external field only acts in the *x* direction (we restrict ourselves therefore to $m=2$). The free energy reads

$$\beta F = -\beta\mu(2\sigma+1) + \int \frac{d^d\mathbf{k}}{(2\pi)^d} \ln\left[2\sinh\left(\frac{\beta}{2}[\mu - J(\mathbf{k})]\right)\right] + \ln\left[2\sinh\left(\frac{\beta\mu}{2}\right)\right] - \frac{\beta\Gamma^2}{2\mu} \tag{51}$$

and the saddle point equation

$$2(\sigma+1) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{1-e^{-\beta[\mu - J(\mathbf{k})]}} + \frac{1}{1-e^{-\beta\mu}} + \frac{\Gamma^2}{2\mu^2}, \tag{52}$$

where we have applied the changes $J_\lambda \to J(\mathbf{k})$ and

$$\int dJ_\lambda \rho(J_\lambda) = \int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d}. \tag{53}$$

We choose $J(\mathbf{k}) \approx J_0 - J'|\mathbf{k}|^x$ for $|\mathbf{k}| \to 0$. In the case of short range couplings, for instance, one has $x=2$ since $J(\mathbf{k}) = \Sigma J \cos k_i \approx J(0) - \frac{1}{2}J|\mathbf{k}|^2$. A long range coupling that decays as $J(r) \sim 1/r^{-\alpha}$ at large r gives $x = \alpha - d$.

As in the theory of Bose-Einstein condensation, the saddle point equation fixes the dependence of μ on temperature. There should be a solution at any T. In order to have a real free energy, μ cannot be smaller than the maximum value for $J(\mathbf{k})$. Therefore, we should investigate the convergence of the integral in the limit $\mu \to J_0$. If the integral diverges, β must go to infinity before μ reaches J_0 in order to satisfy the saddle point equation, so there exists a μ for all temperatures and no phase transition occurs. If the integral converges, however, there will be a range of temperatures in which the saddle point as it stands cannot hold. This indicates that we have overlooked a macroscopic occupation of the ground state, as occurs in Bose-Einstein condensation. The relevant integral behaves as

$$\int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{1-e^{-\beta[J_0 - J(\mathbf{k})]}} \approx \frac{\Omega_d}{(2\pi)^d} \int_0 dk k^{d-1} \frac{1}{1-e^{-\beta J' k^x}} \propto \int_0 dk k^{d-1-x}, \tag{54}$$

where Ω_d is the hypersurface of a sphere in d dimensions. At $\mathbf{k}=0$, this integral converges for $d > x$, hence there will be a phase transition for dimensions larger than x.

At low temperatures, μ may get stuck at J_0 and the saddle point equation as it is in Eq. (52) is no longer valid. This is because, as in Bose-Einstein condensation calculations, the ground state is not properly included in the integral. It should be taken out of the sum before this one is converted to an integral. This causes a change in the free energy by a factor $(\mu - J_0)q$ and the saddle point equation becomes

$$2(\sigma+1) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{1-e^{-\beta[\mu - J(\mathbf{k})]}} + \frac{1}{1-e^{-\beta\mu}} + \frac{\Gamma^2}{2\mu^2} + q, \tag{55}$$

where $q = 1/N\langle \hat{\Sigma}^{z\dagger}_{k=0} \hat{\Sigma}^z_{k=0}\rangle$ is the ground state occupation. q can be evaluated from the saddle point equation $(\mu - J_0)\sqrt{q} = 0$. Thus when $\mu = J_0$ the occupation of the ground state can take nonzero values that can be determined using Eq. (55). Hence the ground state occupation is macroscopic in the ordered phase.

A transversal field will lower the transition temperature. Above a certain value Γ_c, the transition does not exist anymore, thus $T=0$, $\Gamma = \Gamma_c$ is a quantum critical point (for a complete study over quantum phase transitions, see, e.g. Ref. [19]). We will now first study the classical critical point, where $\Gamma = 0$.

1. Finite temperature phase transition

For the dimensions where the phase transition exists, the critical temperature is found by solving the equation

$$2(\sigma+1) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{1-e^{-\beta_c[J_0 - J(\mathbf{k})]}} + \frac{1}{1-e^{-\beta_c J_0}} \tag{56}$$

The dependence of the chemical potential on the temperature near the transition is the first thing needed. To get it, we expand the saddle point equation around the critical point $T = T_c + \tau$, $\mu = J_0 + \delta\mu$. The integral gives, up to first order in $\delta\mu$ and τ

$$\int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{1-e^{-\beta(\mu - J(\mathbf{k}))}} \approx \int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \left[\frac{1}{1-e^{-\beta_c[J_0 - J(\mathbf{k})]}} + \tau \frac{J_0 - J(\mathbf{k})}{4T_c^2 \sinh^2\left(\frac{J_0 - J(\mathbf{k})}{2T_c}\right)} - \delta\mu \frac{1}{4T_c \sinh^2\left(\frac{J_0 - J(\mathbf{k})}{2T_c}\right)} \right]. \tag{57}$$

The coefficient of $\delta\mu$ is an integral that diverges for $d \leqslant 2x$. This means that for these dimensions the leading term in the $\delta\mu$ expansion of Eq. (57) has a power smaller than one. For dimensions $d > 2x$ we will have $\delta\mu \propto \tau$ which will lead to the mean-field exponents, $2x$ is therefore the upper critical dimension. To study the system near the critical point we subtract Eq. (56) from the saddle point equation, a procedure that will cancel the zeroth order term in the expansion in τ and $\delta\mu$, giving finally

$$\tau \approx a_{d<2x} \delta\mu^{(d-x)/x} \quad \text{for} \quad x < d < 2x,$$

$$a_{d<2x} = \alpha \frac{4\Omega_d T_c^3 \pi}{(2\pi)^d J'^{\frac{d}{x}} x \sin\left(\frac{(d-x)\pi}{x}\right)},$$

$$\tau \approx a_{d=2x}\delta\mu \ln \delta\mu \quad \text{for} \quad d=2x,$$

$$a_{d=2x}=\alpha\frac{4\Omega_d T_c^3}{(2\pi)^d J'^2 x},$$

$$\tau \approx a_{d>2x}\delta\mu \quad \text{for} \quad d>2x,$$

$$a_{d>2x}=\alpha T_c\left[\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{\sinh^2\left(\frac{J_0-J(\mathbf{k})}{2T_c}\right)}+\frac{1}{\sinh^2\left(\frac{J_0}{2T_c}\right)}\right] \tag{58}$$

where

$$\alpha=\left[\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{J_0-J(\mathbf{k})}{\sinh^2\left(\frac{J_0-J(\mathbf{k})}{2T_c}\right)}+\frac{J_0}{\sinh^2\left(\frac{J_0}{2T_c}\right)}\right]^{-1} \tag{59}$$

is a finite, positive number.

The internal energy of the system reads

$$U=-\mu(2\sigma+1)+\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{\mu-J(\mathbf{k})}{2}\coth\left[\frac{\beta[\mu-J(\mathbf{k})]}{2}\right]+\frac{\mu}{2}\coth\left(\frac{\beta\mu}{2}\right)-\frac{\Gamma^2}{2\mu}. \tag{60}$$

The specific heat close to the transition from the paramagnetic side can be written as

$$C\approx\begin{cases} C_0+\dfrac{x}{a_{d<2x}(d-x)}C_1\tau^{(2x-d)/(d-x)} & \text{for} \quad x<d<2x \\ C_0+\dfrac{1}{a_{d>2x}}C_1 & \text{for} \quad d>2x \end{cases} \tag{61}$$

where

$$C_0=\frac{1}{4T^2}\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{[\mu-J(\mathbf{k})]^2}{\sinh^2\left(\frac{\mu-J(\mathbf{k})}{2T}\right)}+\frac{\mu^2}{4T^2\sinh^2\left(\frac{\mu}{2T}\right)}, \tag{62}$$

$$C_1=-2\sigma-1+\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\left\{\frac{1}{2}\coth\left[\frac{\mu-J(\mathbf{k})}{2T}\right]-\frac{\mu-J(\mathbf{k})}{4T\sinh^2\left(\frac{\mu-J(\mathbf{k})}{2T}\right)}\right\}+\frac{1}{2}\coth\left(\frac{\mu}{2T}\right)-\frac{\mu}{4T\sinh^2\left(\frac{\mu}{2T}\right)}+\frac{\Gamma^2}{2\mu^2}, \tag{63}$$

where a_d are the prefactors in Eq. (58) for the corresponding dimension. In the ordered phase μ is stuck in its minimum value ($\mu=J_0$) for any temperature. Hence, $C=C_0(\mu=J_0)$ in the ordered phase. The critical exponent α is the expected one: $\alpha=(d-2x)/(d-x)$ for $x<d<2x$, and the mean-field value $\alpha=0$ holds for $d>2x$, which describes a jump in the specific heat.

Adding a small longitudinal field h, the free energy reads

$$\beta F=-\beta\mu(2\sigma+1)+\int\frac{d^d\mathbf{k}}{(2\pi)^d}\ln\left[2\sinh\left(\frac{\beta}{2}[\mu-J(\mathbf{k})]\right)\right]+\ln\left[2\sinh\left(\frac{\beta\mu}{2}\right)\right]-\frac{\beta\Gamma^2}{2\mu}-\frac{\beta h^2}{2(\mu-J_0)} \tag{64}$$

and the saddle point equation becomes

$$2(\sigma+1)=\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{1-e^{-\beta(\mu-J(\mathbf{k}))}}+\frac{1}{1-e^{-\beta\mu}}+\frac{\Gamma^2}{2\mu^2}+\frac{h^2}{2(\mu-J_0)^2}. \tag{65}$$

By differentiating the free energy with respect to h it can be seen that the magnetization is $M_z=h/(\mu-J_0)$. In the limit $h\to 0$, it is proportional to the square root of the occupation of the ground state, since by comparing Eq. (65) with Eq. (55) one finds $q=(1/N)\langle\hat{\Sigma}_{k=0}^{z\dagger}\hat{\Sigma}_{k=0}^{z}\rangle=M_z^2/2$. The factor $\frac{1}{2}$ appears because it is actually the real part of the spin field the one macroscopically occupied and a half term appears in the change Eq. (6). From Eq. (65), we can approach the transition by sending the longitudinal field to zero at the critical temperature. The saddle point equation now accounts for the dependence of the chemical potential on the field. The calculation is similar, yielding finally

$$h\approx\left(\frac{2\Omega_d T_c\pi}{(2\pi)^d J'^{d/x}x\sin\left(\frac{\pi(d-x)}{x}\right)}\right)^{1/2}\delta\mu^{(d+x)/2x}$$

$$\text{for} \quad x<d<2x,$$

$$h\approx\left[\frac{2\Omega_d T_c}{(2\pi)^d J'^2 x}\ln\delta\mu\right]^{1/2}\delta\mu^{3/2} \quad \text{for} \quad d=2x,$$

$$h\approx\left(\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{2T_c\sinh^2\left(\frac{J_0-J(\mathbf{k})}{2T_c}\right)}+\frac{1}{2T_c\sinh^2\left(\frac{J_0}{2T_c}\right)}\right)^{1/2}\delta\mu^{3/2} \quad \text{for} \quad d>2x. \tag{66}$$

Therefore the critical exponent δ is given by $\delta=(d+x)/(d-x)$ for dimensions $x<d<2x$ and the mean-field value $\delta=3$ is recovered for $d>2x$. From the magnetization, the susceptibility follows as $\chi \approx 1/\delta\mu$. Therefore we find $\gamma = x/(d-x)$ for $x<d<2x$ and $\gamma=1$ for $d>2x$. In the ordered phase, the expansion of the saddle point equation, Eq. (65), for T near the transition yields

$$M_z^2 \approx \tau \frac{1}{2T_c^2}\left[\int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{J_0 - J(\mathbf{k})}{\sinh^2\left(\frac{J_0-J(\mathbf{k})}{2T_c}\right)} + \frac{J_0}{\sinh^2\left(\frac{J_0}{2T_c}\right)}\right]. \tag{67}$$

Therefore for all dimensions where the phase transition exists, one has $\beta = \frac{1}{2}$.

For other critical exponents the correlation function is needed. It can be computed adding the right source term to the Hamiltonian, $\Sigma_\lambda (g_\lambda(\tau_q)\hat{\mathbf{\Sigma}}_\lambda^\dagger + g_\lambda^*(\tau_r)\hat{\mathbf{\Sigma}}_\lambda)$ and differentiating

$$\begin{aligned} G(\lambda,\tau_q|\lambda',\tau_r) &= \langle \mathrm{T}[\hat{\Sigma}_\lambda^{z\,\dagger}(\tau_q)\hat{\Sigma}_{\lambda'}^z(\tau_r)]\rangle \\ &= \delta_{\lambda,\lambda'} \frac{\partial^2}{\partial g_\lambda^*(\tau_q)\partial g_{\lambda'}(\tau_r)} \frac{Z(g^*,g)}{Z_0}\bigg|_{g^*=g=0}, \end{aligned} \tag{68}$$

where T stands for the time ordered product. $Z(g^*,g)$ is the partition function of the Hamiltonian including the source terms and Z_0 is the partition function without them. This procedure is carefully explained in Ref. [17] giving the result

$$\begin{aligned} G(k,\tau) &= G(k,\tau|k,0) \\ &= e^{\tau(\mu-J(k))}\{\theta(\tau-\eta)(1+n_k)+\theta(-\tau+\eta)n_k\}, \end{aligned} \tag{69}$$

where θ is a Heaviside step function and η is a positive infinitesimal that indicates that the second term is the relevant one at $\tau=0$. Furthermore,

$$n_k = \frac{1}{e^{\beta\omega}-1} \tag{70}$$

is the boson occupation probability, with $\omega = \mu - J(\mathbf{k})$. Fourier transforming this last result to Matsubara frequencies we get

$$G(k,\omega_n) = \frac{1}{J(k)-\mu-i\omega_n} \xrightarrow{k\to 0} \frac{-1}{J'|k|^x+\delta\mu+i\omega_n}. \tag{71}$$

So when we approach the critical point, we can see from this equation that $\xi^{-x}\propto\delta\mu$, then using Eq. (58) we find that $\nu = 1/(d-x)$ for dimensions $x<d<2x$ and $\nu=1/x$ for $d>2x$. $\eta = 2-x$ due to the fact that the couplings depend on k^x, and $z=x$ because in the denominator ω appears as a linear term. Both η and z are valid for any dimension.

This finally gives all the critical exponents of this finite temperature phase transition, which are exactly the same as in the classical model. This is expected from renormalization group arguments [16]. The critical behavior is controlled by a classical fixed point, therefore quantum dynamics does not play a qualitatively new role. Hence, the results are the same as in the classical spherical model [3] or other models with different quantum dynamics considered at finite temperatures [13].

2. T=0 quantum phase transition

In this section we analyze the behavior of the system at $T=0$. As it can be seen from Eq. (52), when the transversal field increases, the temperature of the transition decreases till it reaches zero. This defines a quantum critical point $T_c=0$ at $\Gamma=\Gamma_c$. In order to study it, an analogous procedure as before should be followed. At $T=0$ everything happens to be rather simple. The free energy reduces to

$$F = -2\sigma\mu - \frac{\Gamma^2}{2\mu}. \tag{72}$$

The saddle point equation turns out to be

$$2\sigma = \frac{\Gamma^2}{2\mu^2} \quad \text{in the paramagnetic phase,}$$

$$2\sigma = \frac{\Gamma^2}{2J_0^2} + q \quad \text{in the ferromagnetic phase.} \tag{73}$$

where $q = \frac{1}{2}M_z^2$ is the occupation of the ground state for small transversal fields. Since the temperature vanishes, quantum fluctuations, controlled by Γ, give rise to the phase transition. Therefore, the parameter that should be used to control the transition is the transversal field and not the temperature. Then the proper analog of the specific heat will be proportional to the second derivative of the free energy with respect to the source of fluctuations, the transversal field.

$$C_\Gamma \equiv \frac{\partial^2 F}{\partial\Gamma^2} = -\frac{1}{\mu} + \frac{\Gamma}{\mu^2}\frac{\partial\mu}{\partial\Gamma}. \tag{74}$$

As before, we must know the dependence of $\delta\mu$ ($\mu = J_0 + \delta\mu$) on the distance to the critical point ($\delta\Gamma$) in the paramagnetic phase. In this problem, the lower critical dimension is $d_{lc}=0$, since for $d>0$ the volume element $\int d^d\mathbf{k}$ is finite. The upper critical dimension will be $d_{uc}=x$. Between those two dimensions, $0<d<x$, the analysis of the saddle point equation yields that the product $\beta\delta\mu$ goes to finite, strictly positive value for $T\to 0$. This leads to a scaling from $\delta\mu \propto \delta\Gamma^{x/d}$. On the $T=0$ line, the analog of the specific heat goes continuously from the paramagnetic vaule $C_\Gamma \approx -1/J_0 + \Gamma_c/J_0^2\delta\Gamma^{(x-d)/d}$ to the simple ferromagnetic value $C_\Gamma = -1/J_0$. This implies that $\alpha = (d-x)/d$. Adding a longitudinal field we find the dependence $\delta\mu \propto \delta h^{2x/(d+2x)}$ bringing $\delta = (d+2x)/d$ and $\gamma = d/x$. Subtracting the saddle point equation near the transition in the ferromagnetic phase from the one at the transition, we get $q = (\Gamma^2 - \Gamma_c^2)/2J_0^2$, which in the lowest order gives $M_z^2 \propto \delta\Gamma$ and therefore, as

always, $\beta=\frac{1}{2}$. Equation (71) can be used here once it is transformed to real frequencies, $i\omega_n=\omega+i\eta$. Then we find $\nu=1/d$, $\eta=2-x$, and $z=x$.

For dimensions $d>x$, from Eqs. (74) and (73), it can be seen that the analog of the specific heat has a jump discontinuity, implying $\alpha=0$.

$$C_\Gamma=0 \quad \text{in the paramagnetic phase,}$$

$$C_\Gamma=\frac{-1}{J_0} \quad \text{in the ferromagnetic phase.} \tag{75}$$

where the minus sign comes from the fact that the $T=0$ free energy, Eq. (72), is negative. Adding a small longitudinal field as before, we find the critical exponent $\delta=3$, since $M_z\propto h/\delta\mu$ and $\delta\mu\propto h^{2/3}$ at Γ_c. For the susceptibility, we find $\gamma=1$ since $\chi\propto\delta\mu^{-1}\propto\delta\Gamma^{-1}$. As before we find $\beta=\frac{1}{2}$ and since $\delta\mu\propto\delta\Gamma$ we find $\nu=1/x$, $\eta=2-x$ and $z=x$. Hence, for $d>x$, we find the mean-field values. This occurs because the quantum critical point of a d-dimensional model shares the critical exponents of a classical critical point of a $(d+z)$-dimensional model, as it was shown by general renormalization group arguments [16].

V. HAMILTONIANS INVOLVING SPINS BUT NOT THEIR MOMENTA

In this section we are going to extend the analysis of Sec. IV to a Hamiltonian which only depends on the spin operators $\hat{S}$ and not on the momenta $\hat{\Pi}$. When going from the classical to the quantum model, we have to keep in mind that the Hamiltonian must be Hermitian. To be precise, the Hamiltonian we will deal with is

$$H=-\frac{1}{2}\sum_{ij} J_{ij}\hat{\mathbf{S}}_i\hat{\mathbf{S}}_j-\sum_i \Gamma_i\hat{\mathbf{S}}_i \tag{76}$$

with real valued J_{ij} and Γ_i and where $\hat{S}=(\hat{\Sigma}^\dagger+\hat{\Sigma})/\sqrt{2}$ is the real part of the former spin field. Hence, the Hamiltonian does not involve momenta, but the spherical constraint does, see Eq. (16). This changes the symmetry of the problem from invariance under unitary transformations to orthogonal ones. In terms of boson creation and annihilation operators the coupling term for symmetric interactions, $J_{ij}=J_{ji}$, is proportional to $J_{ij}(2\hat{\boldsymbol{\Sigma}}_i^\dagger\hat{\boldsymbol{\Sigma}}_j+\hat{\boldsymbol{\Sigma}}_i\hat{\boldsymbol{\Sigma}}_j+\hat{\boldsymbol{\Sigma}}_i^\dagger\hat{\boldsymbol{\Sigma}}_j^\dagger)$, where we can notice the symmetry of the problem. We will see that this model reproduces the $O(\mathcal{N})$ quantum rotor model.

We can get the partition function in many ways. A similar procedure using discrete imaginary time path integrals can be done as before. This gives us many problems due to the fact that creation and annihilation operators are projected on different time steps which is a lengthy and tedious procedure. However, the form of the Hamiltonian makes it suitable to apply a Bogoliubov transformation (for details see, e.g., Ref. [20]). Due to that, we get the same Hamiltonian as before but with different coefficients. In order to do so, we must add the spherical constraint directly to the Hamiltonian. The procedure is as follows: first the couplings matrix is diagonalized by inverting the lattice as done before in Eq. (40) and then the $\hat{S}$'s are shifted to absorb the field term. This finally gives

$$H=\sum_\lambda \left\{\left(\mu-\frac{J_\lambda}{2}\right)\hat{\boldsymbol{\Sigma}}_\lambda^\dagger\hat{\boldsymbol{\Sigma}}_\lambda-\frac{J_\lambda}{4}(\hat{\boldsymbol{\Sigma}}_\lambda^\dagger\hat{\boldsymbol{\Sigma}}_{-\lambda}^\dagger+\hat{\boldsymbol{\Sigma}}_\lambda\hat{\boldsymbol{\Sigma}}_{-\lambda}) -\frac{\Gamma_\lambda^2}{2(\mu-J_\lambda)}\right\}-Nm\mu\sigma. \tag{77}$$

Performing the Bogoliubov transformation it turns into

$$H=\sum_\lambda \left\{\sqrt{\mu(\mu-J_\lambda)}\hat{\alpha}_\lambda^\dagger\hat{\alpha}_\lambda-\frac{\Gamma_\lambda^2}{2(\mu-J_\lambda)}\right\}-Nm\mu\sigma, \tag{78}$$

which is a Hamiltonian analogous to Eq. (39). So it can be diagonalized as explained, giving finally

$$\beta F=-\beta\mu m\left(\sigma+\frac{1}{2}\right)+m\int dJ_\lambda\rho(J_\lambda) \times\left\{\ln\left[2\sinh\left(\frac{\beta}{2}\sqrt{\mu(\mu-J_\lambda)}\right)\right]-\frac{\beta\Gamma_\lambda^2}{2(\mu-J_\lambda)}\right\}, \tag{79}$$

where we have put back the factor m standing for the number of components of the vector spin. The saddle point equation is obtained as

$$\sigma+\frac{1}{2}=\int dJ_\lambda\rho(J_\lambda)\left\{\frac{2\mu-J_\lambda}{4\sqrt{\mu(\mu-J_\lambda)}}\coth\frac{\beta}{2}\sqrt{\mu(\mu-J_\lambda)} +\frac{\beta\Gamma_\lambda^2}{2(\mu-J_\lambda)^2}\right\}. \tag{80}$$

At large temperatures these equations reduce to

$$\beta F=-\beta\mu m\left(\sigma+\frac{1}{2}\right)+m\int dJ_\lambda\rho(J_\lambda)\left\{\ln\beta\sqrt{\mu(\mu-J_\lambda)} -\frac{\beta\Gamma_\lambda^2}{2(\mu-J_\lambda)}\right\}, \tag{81}$$

$$\sigma+\frac{1}{2}=\int dJ_\lambda\rho(J_\lambda)\left\{\frac{T}{2\mu}+\frac{T}{2(\mu-J_\lambda)}+\frac{\beta\Gamma_\lambda^2}{2(\mu-J_\lambda)^2}\right\}, \tag{82}$$

These equations are very similar to the standard ones of the classical spherical model (up to a factor 2), see Eq. (49), but they are only identical where they should be, namely, at large T, where also $\mu \sim T$ is very large, see also Ref. [3].

A. Ferromagnetic couplings in the presence of a transversal field

Analyzing the phase transition of the Hamiltonian that does not depend on the momenta is analogous to the previous case. We begin again by choosing the coupling term in the z direction and the external field in the x and we assume ferromagnetic couplings. The saddle point equation gives a phase transition via a macroscopic occupation of the ground state, which in the present case is a bit more complicated. The critical exponents are different, due to the fact that the symmetries of the system have changed. The free energy reads

$$\beta F = -\beta\mu(2\sigma+1) + \int \frac{d^d\mathbf{k}}{(2\pi)^d} \ln\left[2\sinh\left(\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right)\right] + \ln\left[2\sinh\left(\frac{\beta\mu}{2}\right)\right] - \frac{\beta\Gamma^2}{2\mu} \tag{83}$$

and the saddle point equation

$$4\sigma+2 = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{2\mu-J(\mathbf{k})}{2\sqrt{\mu[\mu-J(\mathbf{k})]}} \coth\left[\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right] + \coth\left(\frac{\beta\mu}{2}\right) + \frac{\Gamma^2}{\mu^2}. \tag{84}$$

We now analyze this model in detail.

1. Finite temperature phase transition

Following the same procedure as before we can find that the transition exists for $d>x$ and that the upper critical dimension is $d=2x$. The critical temperature is the solution of

$$4\sigma+2 = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{2J_0-J(\mathbf{k})}{2\sqrt{J_0[J_0-J(\mathbf{k})]}} \coth\left[\frac{\beta}{2}\sqrt{J_0[J_0-J(\mathbf{k})]}\right] + \coth\left(\frac{\beta J_0}{2}\right). \tag{85}$$

The dependence of the chemical potential in the temperature near the classical critical point reads

$$\tau \approx a_{d<2x}\delta\mu^{(d-x)/x} \quad \text{for} \quad x<d<2x,$$

$$a_{d<2x} = \alpha \frac{2\Omega_d T_c^3 \pi}{(2\pi)^d J'^{\frac{d}{x}} x \sin\left(\frac{(d-x)\pi}{x}\right)}$$

$$\tau \approx a_{d=2x}\delta\mu \ln\delta\mu, \quad a_{d=2x} = \alpha\frac{2\Omega_d T_c^3}{(2\pi)^d J'^2 x} \quad \text{for} \quad d=2x$$

$$\tau \approx a_{d>2x}\delta\mu, \quad \text{for} \quad d>2x$$

$$a_{d>2x} = \alpha\left(\frac{\partial}{\partial\mu}\left\{\int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{2\mu-J(\mathbf{k})}{2\sqrt{\mu(\mu-J(\mathbf{k}))}} \times \coth\left[\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right] + \coth\left(\frac{\beta\mu}{2}\right)\right\}\right)_{\mu=J_0}, \tag{86}$$

where

$$\alpha = \left[\int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{2J_0-J(\mathbf{k})}{2\sinh^2\left(\frac{\sqrt{J_0[J_0-J(\mathbf{k})]}}{2T_c}\right)} + \frac{J_0}{\sinh^2\left(\frac{J_0}{2T_c}\right)}\right]^{-1}. \tag{87}$$

The internal energy of the system reads

$$U = -\mu(2\sigma+1) + \int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\sqrt{\mu[\mu-J(\mathbf{k})]}}{2} \coth\left[\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right] + \frac{\mu}{2}\coth\left(\frac{\beta\mu}{2}\right) - \frac{\Gamma^2}{2\mu}. \tag{88}$$

The specific heat has the same expression as in Eq. (61) where a_d now correspond to the prefactors of Eq. (86) and with coefficients

$$C_0 = \frac{1}{4T^2}\int_{-\pi}^{\pi} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{\mu[\mu-J(\mathbf{k})]}{\sinh^2\left(\frac{\sqrt{\mu[\mu-J(\mathbf{k})]}}{2T}\right)} + \frac{\mu^2}{4T^2\sinh^2\left(\frac{\mu}{2T}\right)} \tag{89}$$

$$C_1=-(2\sigma+1)+\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\left\{\frac{2\mu-J(\mathbf{k})}{4\sqrt{\mu[\mu-J(\mathbf{k})]}}\times\coth\left(\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right)-\frac{2\mu-J(\mathbf{k})}{8T\sinh^2\left(\frac{\sqrt{\mu[\mu-J(\mathbf{k})]}}{2T}\right)}\right\}+\frac{1}{2}\coth\left(\frac{\mu}{2T}\right)-\frac{\mu}{4T\sinh^2\left(\frac{\mu}{2T}\right)} \quad (90)$$

This is analogous to the previous model and gives the same exponent, $\alpha=(d-2x)/(d-x)$ for $x<d<2x$ and $\alpha=0$ for $d>2x$. Adding a small magnetic field longitudinal to the couplings, the free energy becomes

$$\beta F=-\beta\mu(2\sigma+1)+\int\frac{d^d\mathbf{k}}{(2\pi)^d}\ln\left[2\sinh\left(\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right)\right]+\ln\left[2\sinh\left(\frac{\beta\mu}{2}\right)\right]-\frac{\beta\Gamma^2}{2\mu}-\frac{\beta h^2}{2(\mu-J_0)} \quad (91)$$

therefore the magnetization is $M_z=h/(\mu-J_0)$, which is as before the square root of the occupation of the ground state $q=1/N\langle\hat{S}_z(|\mathbf{k}|=0)^2\rangle=M_z^2$. The saddle point equation is now

$$2(2\sigma+1)=\int_{-\pi}^{\pi}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{2\mu-J(\mathbf{k})}{2\sqrt{\mu[\mu-J(\mathbf{k})]}}\times\coth\left\{\frac{\beta}{2}\sqrt{\mu[\mu-J(\mathbf{k})]}\right\}+\coth\left(\frac{\beta\mu}{2}\right)+\frac{\Gamma^2}{\mu^2}+\frac{h^2}{(\mu-J_0)^2}. \quad (92)$$

With all these and following the algebra of the preceding section one finds the same critical exponents for the magnetization for the same dimensions since we are in the classical critical point.

The time ordered correlation function $\langle \mathrm{T}\hat{S}_k^z(\tau)\hat{S}_{-k}^z(0)\rangle$ differs from the previous one, Eq. (69), since in this case the $\hat{S}$ are not the variables that diagonalize the Hamiltonian in Eq. (78). We must write $\hat{S}$ in terms of $\hat{\boldsymbol{\alpha}}$ and then compute the correlations. This brings

$$G(k,\tau|k,0)=\frac{J(\mathbf{k})}{4\sqrt{\mu[\mu-J(\mathbf{k})]}}\{n_k\cosh[\tau\sqrt{\mu[\mu-J(\mathbf{k})]}]+e^{-|\tau|\sqrt{\mu[\mu-J(\mathbf{k})]}}\} \quad (93)$$

where

$$n_k=\frac{1}{e^{\beta\sqrt{\mu[\mu-J(\mathbf{k})]}}-1}, \quad (94)$$

which in frequency space reads

$$G(k,i\omega_n)=\frac{-J(\mathbf{k})}{2[\omega_n^2-\mu[\mu-J(\mathbf{k})]]}. \quad (95)$$

When approaching the critical point we find that $\xi^{-x}\propto\delta\mu$ as before and we get the same value $\nu=1/(d-x)$ for dimensions $x<d<2x$ and $\nu=1/x$ for $d>2x$. Since couplings appear in the same way as before we also get the same value, $\eta=2-x$ for all dimensions.. The difference appears in the dynamical critical exponent. Here ω_n appears squared, therefore $z=x/2$. Here we see how the model reproduces the critical exponents of the rotor model as in Ref. [13] bringing thus a different behavior at the $T=0$ quantum critical point from the model of Sec. IV A 2.

2. T=0 quantum phase transition

In this case the $T=0$ phase transition is more interesting due to the fact that the dynamical critical exponent $z=x/2$ is smaller than $z=x$ of the preceding section. The free energy reads

$$F=-\mu(2\sigma+1)+\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{\sqrt{\mu[\mu-J(\mathbf{k})]}}{2}+\frac{\mu}{2}-\frac{\Gamma^2}{2\mu} \quad (96)$$

and the saddle point is set by

$$4\sigma+1=\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{2\mu-J(\mathbf{k})}{2\sqrt{\mu[\mu-J(\mathbf{k})]}}+\frac{\Gamma^2}{\mu^2}. \quad (97)$$

We find that the transition exists for dimensions larger than $d>x/2$ and $d=3x/2$ is the upper critical dimension. The chemical potential depends on the source of fluctuations $\delta\Gamma=\Gamma-\Gamma_c$ as

$$\delta\Gamma \approx a_{d<3x/2}\delta\mu^{2d-x/2x},$$

$$a_{d<3x/2} = \frac{\Omega_d J_0^{5/2}}{2(2\pi)^d J'^{d/x}} \frac{\Gamma\left(\frac{3}{2}-\frac{d}{x}\right)\Gamma\left(\frac{d}{x}\right)}{(2d-x)x\sqrt{\pi}} \quad \text{for} \quad \frac{x}{2}<d<\frac{3x}{2},$$

$$\delta\Gamma \approx a_{d=3x/2}\delta\mu \ln\delta\mu,$$

$$a_{d=3x/2} = \frac{\Omega_d}{(2\pi)^d}\frac{J_0^{5/2}}{8xJ'^{3/2}} \quad \text{for} \quad d=\frac{3x}{2},$$

$$\delta\Gamma \approx a_{d>3x/2}\delta\mu, \quad \text{for} \quad d>\frac{3x}{2}$$

$$a_{d=3x/2} = \frac{J_0^2}{2}\left[\frac{2\Gamma_c^2}{J_0^3} - \int \frac{d^d\mathbf{k}}{(2\pi)^d}\left\{\frac{1}{\sqrt{J_0[J_0-J(\mathbf{k})]}} - \frac{2J_0-J(\mathbf{k})}{4\{J_0[J_0-J(\mathbf{k})]\}^{3/2}}\right\}\right], \tag{98}$$

where the Γ's on the right hand side of the first equality are Euler's Γ functions. The specific heat [see Eq. (74)] coming from the disordered region will behave as

$$C_\Gamma \approx \begin{cases} \dfrac{-1}{J_0} - \dfrac{4\Gamma_c}{a_{d<3x/2}J_0^2(2d-x)}\delta\Gamma^{-2d+3x/2d-x} & \text{for} \quad \dfrac{x}{2}<d<\dfrac{3x}{2} \\ \dfrac{-1}{J_0} + \dfrac{2\Gamma_c}{J_0^4}\left(\dfrac{2\Gamma_c^2}{J_0^3} - a_{d>3x/2}\right)^{-1} & \text{for} \quad d>\dfrac{3x}{2} \end{cases} \tag{99}$$

where a_d is the prefactor in Eq. (98) for the proper dimension. Coming from the ordered region, conversely, $C_\Gamma \approx -1/J_0$. Therefore $\alpha=(2d-3x)/(2d-x)$ for $x/2<d<3x/2$ and $\alpha=0$ for $d>3x/2$.

The dependence of a small longitudinal field on the chemical potential, in case the transversal field is at its critical value, reads

$$h \approx a_{d<3x/2}^{1/2}\delta\mu^{(2d+3x)/4x} \quad \text{for} \quad \frac{x}{2}<d<\frac{3x}{2},$$

$$h \approx [a_{d=3x/2}\ln\delta\mu]^{1/2}\delta\mu^{3/2} \quad \text{for} \quad d=\frac{3x}{2}$$

$$h \approx \left(-\frac{2\Gamma_c^2}{J_0^3} - a_{d>3x/2}\right)^{1/2}\delta\mu^{3/2} \quad \text{for} \quad d>\frac{3x}{2}. \tag{100}$$

From these equations and the ones for the magnetization and the susceptibility we can find that $\delta=(2d+3x)/(2d-x)$ and $\gamma=2x/(2d-x)$ for $x/2<d<3x/2$, while $\delta=3$ and $\gamma=1$ for $d>3x/2$. As before $\beta=1/2$ for every dimension. For the correlation function the calculation is the same as in the finite temperature case, projected into real time, the exponents are $\nu=2/(2d-x)$ for dimensions $x/2<d<3x/2$ and $\nu=1/x$ above the critical dimension, $\eta=2-x$ and $z=x/2$ for all dimensions.

VI. GENERALIZATION AND MAPPING FROM HEISENBERG SPINS

In this section we generalize the two preceding Hamiltonians and we map the Heisenberg model onto the spherical model. In a more compact way, we can write the former Hamiltonians in absence of external field as

$$H = -\sum_{ij}\left(A_{ij}\hat{\Sigma}_i^\dagger\hat{\Sigma}_j + \frac{B_{ij}}{2}[\hat{\Sigma}_i^\dagger\hat{\Sigma}_j^\dagger + \hat{\Sigma}_i\hat{\Sigma}_j]\right). \tag{101}$$

If the matrices A_{ij} and B_{ij} can be diagonalized simultaneously, the techniques from preceding sections can be used. The free energy reads

$$\beta F = -\beta\mu m\left(\sigma+\frac{1}{2}\right) + \frac{m}{N}\sum_\lambda\left[\frac{\beta A_\lambda}{2} + \ln\left\{2\sinh\left(\frac{\beta}{2}\sqrt{(\mu-A_\lambda)^2-B_\lambda^2}\right)\right\}\right], \tag{102}$$

where μ satisfies the saddle point equation

$$\sigma+\frac{1}{2} = \frac{1}{N}\sum_\lambda \frac{\mu-A_\lambda}{2\sqrt{(\mu-A_\lambda)^2-B_\lambda^2}} \times \coth\left\{\frac{\beta}{2}\sqrt{(\mu-A_\lambda)^2-B_\lambda^2}\right\}. \tag{103}$$

The coefficient B_{ij} in Eq. (101) is responsible for a change in the symmetries of the problem. If B_{ij} is zero, the action is symmetric under unitary transformations while if it is nonzero the symmetry is reduced to orthogonal.

The mapping from Heisenberg spins comes as follows. The Hamiltonian can be written in terms of Schwinger bosons [20]. The Schwinger boson transformation for SU(2) spins reads

$$S^{+}=a_1^{\dagger}a_2, \quad S^{-}=a_1a_2^{\dagger}, \quad S^{z}=\tfrac{1}{2}(a_1^{\dagger}a_1-a_2^{\dagger}a_2). \tag{104}$$

This can be generalized to SU($\mathcal{N}$) spins and expand around the large-$\mathcal{N}$ limit [21]. In a path integral formalism for ferromagnetic interactions

$$H=-\frac{1}{2}\sum_{ij} J_{ij}\mathbf{S_i}\cdot\mathbf{S_j}\to-\frac{1}{2N}\sum_{ij,mn} J_{ij}a^{*}_{jm}a_{im}a^{*}_{in}a_{jn}, \tag{105}$$

where i,j represent lattice sites and m,n represent the boson flavor. The Hilbert space spanned by Schwinger bosons is much larger than the one given by Heisenberg spins. The constraint needed to restrict it to the physical Hilbert space is that the number of Schwinger bosons at each site has to be kept fixed $\Sigma_m^{\mathcal{N}}n_m=\mathcal{N}S$. This is inserted into the formalism in the same way as we have done it with the spherical constraint, a Lagrange multiplier $\mu_i(\tau)$ appears. The biquadratic terms can be decoupled by a Hubbard-Stratonovich transformation (see, e.g., Ref. [17]). In the case of a ferromagnet, the transformation at each time step and for each flavor in the path integral reads

$$\begin{aligned}\exp\left\{-\frac{\epsilon}{2N}\sum_{ij} J_{ij}a_j^{*}a_ia_i^{*}a_j\right\}&\\ \propto\int\prod_{i,j} dQ_{ij}\exp\left\{\frac{\epsilon\mathcal{N}}{2}\sum_{i,j} Q_{ij}J_{ij}Q_{ji}\right.&\\ \left.-\frac{\epsilon}{2}\sum_{ij} Q_{ij}J_{ij}a_j^{*}a_i\right\},&\end{aligned} \tag{106}$$

where a field $Q_{ij}(\tau)$ has been generated. In the mean-field approximation, one puts $Q_{ij}(\tau)=Q$ and $\mu_i(\tau)=\mu$. Hence, one gets up to a non interesting constant

$$\begin{aligned}H_{MF}^{FM-B}(\mathcal{N})=\sum_{i,m}\mu a^{\dagger}_{im}a_{im}-Q\sum_{ij,m}J_{ij}a^{\dagger}_{jm}a_{im}+\frac{NQ^2}{2}\sum_{ij}J_{ij}&\\ -\mathcal{N}NS\mu,&\end{aligned} \tag{107}$$

where we have already added the Schwinger boson constraint. The free energy per particle reads

$$\beta F=\frac{\mathcal{N}}{N}\sum_{\mathbf{k}}\ln(1-e^{-\beta(\mu-QJ(\mathbf{k}))})+\frac{\beta\mathcal{N}Q^2}{2}J(\mathbf{k}=0)-\beta S\mu\mathcal{N} \tag{108}$$

and the saddle point equations are

$$\frac{1}{N}\sum_{\mathbf{k}} n_{\mathbf{k}}=S \tag{109}$$

$$\frac{1}{N}\sum_{\mathbf{k}} J(\mathbf{k})n_{\mathbf{k}}=QJ(\mathbf{k}=\mathbf{0}) \tag{110}$$

where n_k is the boson occupation number, Eq. (70), with $\omega=\mu-QJ(\mathbf{k})$. Subtracting the two saddle point equations we can see that for large S and small T we can approximate $Q\approx S$ recovering then the spherical model Eqs. (46) and (47) for zero external field or Eq. (102) and (103) for $B_{ij}=0$. From this approach we thus see that the free energy of a SU($\mathcal{N}$) Heisenberg ferromagnet for large $\mathcal{N}$ is formally the same as the quantum spherical model proposed in Eq. (39) in the thermodynamic limit, so when the radius of the hypersphere that defines the model [N in Eq. (16)] is also very large. Thus the large $\mathcal{N}$ limit is somehow analogous to Stanley's large spin dimensionality limit.

In the case of an SU($\mathcal{N}$) antiferromagnet the procedure is more or less the same but the symmetries are different. The lattice is divided in two sublattices A,B. In one of the sublattices a spin rotation is performed that allows us to write the Hamiltonian in the form [21]

$$H=\frac{1}{2}\sum_{ij} J_{ij}S_i\cdot S_j\to-\frac{1}{2N}\sum_{ij,mn} J_{ij}a^{*}_{im}a^{*}_{im}a_{jn}a_{jn}. \tag{111}$$

Performing a Hubbard-Stratonovich transformation as before, the Hamiltonian with the Schwinger boson constraint in the mean-field approximation finally reads

$$\begin{aligned}H_{MF}^{AFM-B}(\mathcal{N})=\sum_{i,m}\mu a^{\dagger}_{im}a_{im}-\frac{Q}{2}\sum_{ij,m}J_{ij}(a^{\dagger}_{im}a^{\dagger}_{jm}+a_{im}a_{jm})&\\ +\frac{NQ^2}{2}\sum_{ij}J_{ij}-\mathcal{N}NS\mu.&\end{aligned} \tag{112}$$

It is important to stress that here the SU($\mathcal{N}$) symmetry has been reduced to a residual O($\mathcal{N}$). The free energy per particle reads

$$\begin{aligned}\beta F=\frac{\mathcal{N}}{N}\sum_{\mathbf{k}}\ln\left(2\sinh\left[\frac{\beta}{2}\sqrt{\mu^2-Q^2J^2(\mathbf{k})}\right]\right)-\beta\mathcal{N}\left(S+\frac{1}{2}\right)\mu&\\ +\frac{\beta\mathcal{N}Q^2}{2}J(\mathbf{k}=0)&\end{aligned} \tag{113}$$

and the saddle point equations read

$$\frac{1}{N}\sum_{\mathbf{k}}\frac{\mu}{\sqrt{\mu^2-Q^2J^2(\mathbf{k})}}\left(n_{\mathbf{k}}+\frac{1}{2}\right)=S+\frac{1}{2} \tag{114}$$

$$\frac{1}{N}\sum_{\mathbf{k}}\frac{J^2(\mathbf{k})Q}{\sqrt{\mu^2-Q^2J^2(\mathbf{k})}}\left(n_{\mathbf{k}}+\frac{1}{2}\right)=QJ(\mathbf{k}=\mathbf{0}), \tag{115}$$

where n_k is, Eq. (70), for $\omega=\sqrt{\mu^2-Q^2J^2(\mathbf{k})}$. Subtracting the first equation times μ from the second times Q we get

$$\frac{1}{N}\sum_{\mathbf{k}}\sqrt{\mu^2-Q^2J^2(\mathbf{k})}\left(n_{\mathbf{k}}-\frac{1}{2}\right)=\mu\left(S+\frac{1}{2}\right)-Q^2J(\mathbf{k}=\mathbf{0}). \tag{116}$$

The first term is proportional to T, so for very small temperatures and very large S, near the transition where $\mu\approx QJ(\mathbf{k}=0)$, we can approximate $Q\approx S+\frac{1}{2}$. Then Eqs. (113) and (114) are analogous to Eqs. (102) and (103) for $A_{ij}=0$. This will have the same critical behavior as the model in Sec. V due to the fact that it comes from the term $\coth[\sqrt{\mu-J(\mathbf{k})}]/\sqrt{\mu-J(\mathbf{k})}$ which also appears here due to the equality $2n_{\mathbf{k}}+1=\coth[\sqrt{(\mu+QJ(\mathbf{k}))(\mu-QJ(\mathbf{k}))}]$.

VII. CONCLUSION

In this paper we have explained a way of working with quantum spherical spin models using path integrals and coherent states. Some examples of the use of this formalism are given, Eqs. (39) and (76), and their critical phenomena are studied. We propose a comparison with SU($\mathcal{N}$) Heisenberg models that gives a geometrical interpretation to the quantum spherical spins. The spherical constraint we use, fixes the number of spin quanta $\hat{\Sigma}$, Eq. (16); in other words, it fixes both the average length square of the spin operator, $\hat{S}^2$, and the one of its conjugate momentum, $\hat{\Pi}^2$. The usual version of the quantum spherical model, on the contrary, involves only the spin part $\hat{S}$. The presence of momenta in the spherical constraint allows the Hamiltonian to have no kinetic term, since it can be induced by the constraint, a fact that can change the symmetries of the problem, and due to that, the critical behavior.

The Hamiltonian in Eq. (39) yields an action invariant under unitary transformations. It brings formally the same free energy as a SU($\mathcal{N}$) Heisenberg ferromagnet in the limit of large $\mathcal{N}$. The other Hamiltonian studied, Eq. (76), brings an action invariant under orthogonal transformations; it gives the same critical behavior as an SU($\mathcal{N}$) Heisenberg antiferromagnet in the limit of large $\mathcal{N}$, which is, in its turn, analogous to an O($\mathcal{N}$) nonlinear σ model or quantum rotor model [13,19]. The main difference between these models lies in the dynamical critical exponent z which brings a different behavior at the quantum critical point. Classical critical phenomena are, as expected, the same in both models and equal to those of the classical spherical model.

In the formulation of the model, the strict spherical constraint has been used where fluctuations on the particle number are not allowed. The constraint is added to the action via a Lagrange multiplier. The strict approach has to be abandoned when we integrate this Lagrange multiplier using the saddle point approximation. In this step, we automatically allow fluctuations on the particle number and therefore the constraint ends being satisfied only in average. These effects are immaterial in the considered thermodynamic limit, but do enter finite size corrections.

The analogy of the two Hamiltonians studied here with Heisenberg models in the large spin dimensionality limit has a drawback. Both models have different coupling to the external field. In spherical models it comes in linearly, as a source term. No analog has been found for this in the large spin dimensionality limit of the Heisenberg model where each spin contribution brings a bilinear term in Schwinger bosons.

Another approach could have been to start directly from the SU($\mathcal{N}$) Heisenberg model and to do the already stated large $\mathcal{N}$ limit to get to a solvable model. In order to have a transversal field that competes with the ordering of the interacting spins one could introduce anisotropy in the model. A study of this type has been done for two dimensions by Timm *et al.* [22] in terms of Schwinger bosons and in terms of Holstein-Primakoff bosons by Kaganov *et al.* [23] for any dimension. The anisotropy term brings a residual spin symmetry describing Ising or *XY* spins. The phase transition depends on the type of this residual symmetry; an additional transversal field decreases the transition temperature towards zero giving a quantum critical point, result qualitatively reproduced by our model.

In spite of the lack of direct interpretation of the source term in the mapping from Heisenberg spins, the phase diagram follows the expected behavior for a spin model with an external transversal field. The critical exponents for the classical and the quantum model are the ones expected by renormalization group arguments. The quantum critical point behaves as the classical one for dimensions $D_{quant}=d_{class}+z$ where z is the dynamical critical exponent.

Many other models have a clear analog with this one. Sachdev and Bhatt [24] represented pairs of spins in a square lattice with a bond representation; they form either a singlet or a triplet. These elements can be written down in terms of the canonical "Schwinger boson" representation of the generators of $SU(2)\otimes SU(2)=SO(4)$. Since a couple of spins either form a singlet or a triplet, a constraint must be added $s^\dagger s+\Sigma_\alpha t_\alpha^\dagger t_\alpha=1$, where s represents the singlet annihilation operator, and t_α represents a triplet annihilation operator in the α direction. Sachdev and Bhatt study using this formalism systems with interactions up to third nearest neighbors. They make the further assumption that the singlet part condenses and replace the s operator by its mean field value $\langle s\rangle=\overline{s}$, and solve the rest for the triplets. The final Hamiltonian is very close to our Eq. (76), or, better, the generalization of our model Eq. (101) with the proper couplings. A minor difference is the role played by the nonconstant mean value of the singlet part.

An interesting line for future research would be to expand this version of quantum spherical spin models to different types of interactions and fields. Randomness is easily added in the model. The dynamics could also be studied for the constraint described here. Another line would be to study in this approximation the Heisenberg SU($\mathcal{N}$) model for any dimension with anisotropy in one dimension and transversal field, giving special attention to critical phenomena and compare it to the Holstein-Primakoff approximation of Ref. [23].

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